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# CONVERSE LYAPUNOV THEOREMS FOR SWITCHED SYSTEMS IN BANACH AND HILBERT SPACES

FALK M. HANTE AND MARIO SIGALOTTI

ABSTRACT. We consider switched systems on Banach and Hilbert spaces governed by strongly continuous one-parameter semigroups of linear evolution operators. We provide necessary and sufficient conditions for their global exponential stability, uniform with respect to the switching signal, in terms of the existence of a Lyapunov function common to all modes.

## 1. INTRODUCTION

It is well known that the existence of a *common Lyapunov function* is necessary and sufficient for the global uniform asymptotic stability of finite-dimensional continuous-time switched dynamical systems [14]. In the *linear* finite-dimensional case, the existence of a common Lyapunov function is actually equivalent to global uniform *exponential* stability [17, 18] and, provided that the system has finitely many modes, the Lyapunov function can be taken polyhedral or polynomial (see [2, 3, 8] and also [5] for a discrete-time version). A special role in the switched control literature has been played by common quadratic Lyapunov functions, since their existence can be tested rather efficiently (see the surveys [13, 22] and the references therein). It is known, however, that the existence of a common quadratic Lyapunov function is not necessary for the global uniform exponential stability of a linear switched system with finitely many modes. Moreover, there exists no uniform upper bound on the minimal degree of a common polynomial Lyapunov function [15].

The scope of this paper is to prove that the existence of a common Lyapunov function is equivalent to the global uniform exponential stability of infinite-dimensional switched systems of the type

$$\begin{cases} \frac{d}{dt}x(t) = A_{\sigma(t)}x(t), & t > 0 \\ x(0) = x \in X \end{cases} \quad (1)$$

where each  $A_j$  is a (possibly unbounded) operator generating a strongly continuous semigroup  $T_j(t)$  on a Banach space  $X$  and  $\sigma(\cdot)$  belongs to the class of piecewise constant switching signals with values in an index set  $Q$ .

Such systems provide a convenient design paradigm for modeling a wide variety of complex processes comprising distributed parameters, see [11] and the references therein for examples in the context of networked transport systems.

Except for special cases, that is, when  $X$  has a Hilbert structure and the infinitesimal generators  $A_j$  commute pairwise [21], when the switching signals satisfy a dwell-time constraint [16] or when  $A_j$  is a linear convection-reaction operator with

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reflecting boundary conditions [1], global uniform exponential stability of systems such as (1) has not been investigated (up to our knowledge).

The characterization of exponential stability for a single linear dynamical system on Banach and Hilbert spaces dates back to Datko [7] and Pazy [19] and has, since then, seen a broad range of applications in control theory for partial differential equations (see, for instance, [24]). However, we recall that exponential stability of all subsystems (with  $\sigma(t) \equiv j$  fixed in (1)) is of course necessary but not sufficient for the global uniform exponential stability with respect to all possible switching laws  $\sigma(\cdot)$ . This is a classical result for the finite dimensional case and we give an infinite dimensional variant with interesting destabilizing properties in Example 1 below.

Our starting point will be a switching system of the general form

$$\begin{cases} x(t_{k+1}) = T_{\sigma(t_k)}(t_{k+1} - t_k)x(t_k), & k \in \mathbb{N}, \\ x(0) = x_0 \in X, \end{cases} \quad (2)$$

where  $\sigma: [0, \infty) \rightarrow Q$  is a piecewise constant right-continuous switching signal with switching times  $0 = t_0 < t_1 < \dots < t_k < \dots$ . Each  $t \mapsto T_j(t)$ ,  $j \in Q$ , is a strongly continuous semigroup on a Banach space  $X$ . If  $t \in (t_k, t_{k+1})$ , then  $x(t) = T_{\sigma(t_k)}(t - t_k)x(t_k)$ . This framework includes, in particular, switched dynamical systems such as (1), even with the infinitesimal generators  $A_j$  not sharing a common domain and also in the case of infinitely many available modes  $Q$ . (The semigroup formulation (2) corresponds to the choice of mild solutions for the Cauchy problem (1). The two formulations are clearly equivalent when  $X$  has finite dimension, with  $T_j(t) = e^{tA_j}$ .)

The main result of this paper is that the following three conditions are equivalent:

- (A) *There exist two constants  $K \geq 1$  and  $\mu > 0$  such that, for every  $\sigma(\cdot)$  and every  $x_0$ , the solution  $x(\cdot)$  to (2) satisfies*

$$\|x(t)\|_X \leq K e^{-\mu t} \|x_0\|_X, \quad t \geq 0. \quad (3)$$

- (B) *There exist two constants  $M \geq 1$  and  $\omega > 0$  such that, for every  $\sigma(\cdot)$  and every  $x_0$ , the solution  $x(\cdot)$  to (2) satisfies*

$$\|x(t)\|_X \leq M e^{\omega t} \|x_0\|_X, \quad t \geq 0, \quad (4)$$

*and there exists  $V: X \rightarrow [0, \infty)$  such that  $\sqrt{V(\cdot)}$  is a norm on  $X$ ,*

$$V(x) \leq C \|x\|_X^2, \quad x \in X \quad (5)$$

*for a constant  $C > 0$  and*

$$\liminf_{t \downarrow 0} \frac{V(T_j(t)x) - V(x)}{t} \leq -\|x\|_X^2, \quad j \in Q, \quad x \in X. \quad (6)$$

- (C) *There exists  $V: X \rightarrow [0, \infty)$  such that  $\sqrt{V(\cdot)}$  is a norm on  $X$ ,*

$$c \|x\|_X^2 \leq V(x) \leq C \|x\|_X^2, \quad x \in X \quad (7)$$

*for some constants  $c, C > 0$  and*

$$\liminf_{t \downarrow 0} \frac{V(T_j(t)x) - V(x)}{t} \leq -\|x\|_X^2, \quad j \in Q, \quad x \in X. \quad (8)$$

The equivalence between (A) and (C) extends to infinite-dimensional systems the well-known result obtained in [17] in the finite-dimensional setting.

Conditions (5) and (7) are redundant in the case of finite-dimensional systems, since  $\sqrt{V(\cdot)}$  and  $\|\cdot\|_X$  are comparable, by compactness of the unit sphere. Hence, condition (4) in (B) could be dropped for finite-dimensional systems. This is not the case for infinite-dimensional ones, as illustrated in Remark 4 by an example.

From the point of view of applications, condition (B), imposing less conditions on  $V$  than (C), is better suited for establishing that (A) holds (although the uniform exponential growth boundedness needs also be proved). On the other hand, the implication (A) $\Rightarrow$ (C) can be used to select a Lyapunov function with tighter requirements.

The construction of a common Lyapunov function satisfying (B), under the assumption that (A) holds true, follows the same lines as in finite dimension. In particular, a possible choice of the Lyapunov function is

$$V(x_0) = \sup \left\{ \int_0^\infty \|x(t)\|^2 dt : x(\cdot) \text{ solution to (2) for some } \sigma \right\}.$$

(Alternatively, one could take  $V(x_0) = \int_0^\infty \sup_{\sigma(\cdot)} \|x(t)\|^2 dt$ , as done in [12].)

The construction of a Lyapunov function satisfying (C) is similar, but one has to augment (2) with a further mode  $T_{j^*}(t) = e^{-\mu t}I$  (where  $\mu > 0$  is the constant appearing in (A) and  $I$  denotes the identity on  $X$ ) and to consider all the solutions to this augmented system in the definition of  $V$ .

In the case of an exponentially stable single mode ( $Q = \{0\}$ ), it was observed by Pazy [19] that  $x \mapsto \int_0^\infty \|T_0(t)x\|^2 dt$  defines a Lyapunov function that is comparable with the squared norm if and only if  $T_0$  extends to an exponentially stable strongly continuous group. Notice that, as a consequence of the implication (A) $\Rightarrow$ (C), even if  $T_0$  does not admit an extension to a group, a Lyapunov function comparable with the squared norm can still be found (see Remark 7).

Concerning the regularity of the Lyapunov functions obtained through the construction described above, they are always convex and continuous (since  $\sqrt{V(\cdot)}$  is a norm). In the special case in which  $X$  is a separable Hilbert space, we also prove the Fréchet directional differentiability of  $V$  and we establish a characterization of the directional Fréchet derivatives.

The paper is organized as follows. In Section 2 we introduce the main notations and discuss a motivation example. Section 3 provides a first necessary and sufficient condition for global uniform exponential stability in terms of the existence of a common Lyapunov function, namely, the equivalence of (A) and (B) (Theorem 3). We also discuss the possible redundancies of condition (B), showing that (4) cannot be removed (Remark 4). In Section 4 a second converse Lyapunov theorem is proved, establishing that (A) and (C) are equivalent (Theorem 6). In Section 5 we show the Fréchet differentiability of the common Lyapunov functions constructed in the previous sections when  $X$  is a separable Hilbert space (Corollary 9). In Section 6 we give some final remarks and point to open problems.

## 2. NOTATIONS AND PRELIMINARIES

By  $\mathbb{N}$ ,  $\mathbb{Q}$  and  $\mathbb{R}$  we denote the set of natural, rational and real numbers, respectively. Further, let  $X$  be a Banach space,  $\mathcal{L}(X)$  be the space of bounded linear operators on  $X$ ,  $Q$  be a countable set and, for all  $j \in Q$ , let  $t \mapsto T_j(t) \in \mathcal{L}(X)$ ,  $t \geq 0$  be a strongly continuous semigroup.

We wish to investigate the qualitative behavior of

$$x(t) = T_{\sigma(\cdot)}(t)x \tag{9}$$

for  $x \in X$ , where  $\sigma: [0, \infty) \rightarrow Q$  is a piecewise constant switching signal and

$$T_{\sigma(\cdot)}(t) = T_{j_p}(t - \tau_p)T_{j_{p-1}}(\tau_p - \tau_{p-1}) \cdots T_{j_1}(\tau_1) \tag{10}$$

for  $\sigma(\cdot)$  equal to  $j_k$  on  $(\tau_{k-1}, \tau_k)$  for  $k = 1, \dots, p+1$  and

$$0 = \tau_0 < \tau_1 < \cdots < \tau_{p+1} = t.$$

In particular, we wish to study the asymptotic behavior of  $x(t)$  as  $t$  tends to  $+\infty$ , uniformly with respect to the switching law  $\sigma(\cdot)$  in the set  $\Sigma$  of all piecewise constant switching signals. We note that, for any given  $\sigma(\cdot) \in \Sigma$ , the operator  $T_{\sigma(\cdot)}(t) \in \mathcal{L}(X)$  is strongly continuous with respect to  $t$  ( $\lim_{t \downarrow t_0} \|T_{\sigma(\cdot)}(t) - T_{\sigma(\cdot)}(t_0)\| = 0$ ) and satisfies

$$T_{\sigma(\cdot)}(t+s) = T_{\sigma_s(\cdot)}(t)T_{\sigma(\cdot)}(s) \quad (11)$$

for some switching signal  $\sigma_s(\cdot) \in \Sigma$  depending on  $s$ , but in general,  $T_{\sigma(\cdot)}(t)$  does not satisfy the semigroup property, i. e., equation (11) with  $\sigma_s(\cdot)$  replaced by  $\sigma(\cdot)$  (independently of  $s$ ).

For a function  $V: X \rightarrow [0, \infty)$  we define the generalized derivative

$$\underline{L}_j V(x) = \liminf_{t \downarrow 0} \frac{V(T_j(t)x) - V(x)}{t}, \quad (12)$$

noting the possibility that  $|\underline{L}_j V(x)| = \infty$  for some  $x \in X$  and  $j \in Q$ . Further, we call a switched system (9) (completely determined by  $\{T_j\}_{j \in Q}$ ) *globally uniformly exponentially stable* when there exist constants  $K \geq 1$  and  $\mu > 0$  such that

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq K e^{-\mu t}, \quad t \geq 0, \quad \sigma(\cdot)\text{-uniformly}. \quad (13)$$

It is clear that (13) implies

$$\|x(t)\|_X \leq K e^{-\mu t} \|x\|_X, \quad t \geq 0$$

globally for all  $x \in X$  and uniformly for all  $\sigma(\cdot) \in \Sigma$  justifying the terminology. We point out that (13) implies *strong attractivity at the origin*, i. e.,

$$\lim_{t \rightarrow \infty} \|T_{\sigma(\cdot)}(t)x\|_X = 0, \quad x \in X, \quad \sigma(\cdot) \in \Sigma, \quad (14)$$

and *uniform stability*, i. e.,

$$\begin{cases} \text{for all } \varepsilon > 0 \text{ there exists a } \delta > 0, \\ \text{independent of } \sigma(\cdot), \text{ such that } \|x\|_X < \delta \text{ implies} \\ \|T_{\sigma(\cdot)}(t)x\|_X < \varepsilon, \quad t \geq 0, \quad \sigma(\cdot)\text{-uniformly} \end{cases} \quad (15)$$

but that the converse implication is false in general, even for a single mode: As a counterexample it suffices to take the left translation semigroup defined by

$$(T(t)f)(s) := f(s+t)$$

on the Lebesgue space  $X = L^1(\mathbb{R}_+)$ . This is in contrast to the equivalence of (14) and (13) when  $X$  is a  $n$ -dimensional real coordinate space,  $Q$  is finite, and  $T_j(t)$  is given by the matrix exponential  $e^{A_j t}$  for some real  $n \times n$ -matrix  $A_j$ , as a consequence of Fenichel's Uniformity Lemma (see, for instance, [6, §5.2]).

Before turning our attention to necessary and sufficient conditions for global uniform exponential stability, we give an example of a switched system exhibiting illustrative instability properties, though the subsystems are exponentially stable.

*Example 1.* Consider the bimodal system  $\{T_j(t)\}_{j=1,2}$  with  $T_j(t)$  defined on the Lebesgue space  $X = L^1(-1, 1)$  by

$$(T_1(t)f)(s) = \begin{cases} 2f(s+t), & s \in [-1, 1-t] \cap [-t, 0] \\ f(s+t), & s \in [-1, 1-t] \setminus [-t, 0] \\ 0, & s \in (1-t, 1] \end{cases}$$

and

$$(T_2(t)f)(s) = \begin{cases} 2f(s-t), & s \in [-1+t, 1] \cap [0, t] \\ f(s-t), & s \in [-1+t, 1] \setminus [0, t] \\ 0, & s \in [-1, -1+t]. \end{cases}$$

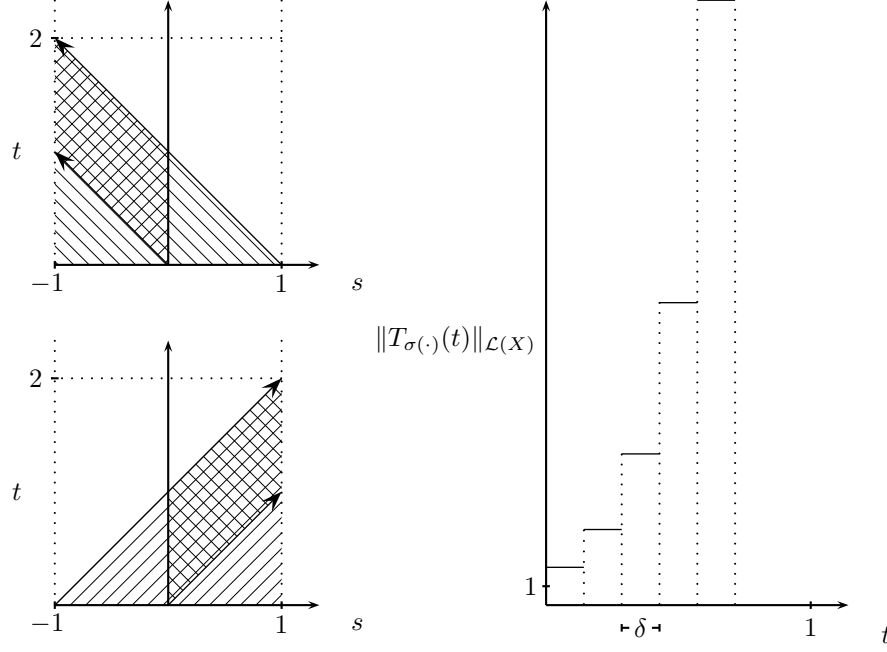


FIGURE 1. Illustration of  $T_j(t)$ ,  $j = 1, 2$ , (left) and the blow up of the operator norm of  $T_{\sigma(\cdot)}(t)$  (right) in Example 1.

Notice that both  $T_1(\cdot)$  and  $T_2(\cdot)$  are nilpotent semigroups, since  $T_1(t) = T_2(t) = 0$  for  $t \geq 2$ . In particular, each of them is exponentially stable.

It is easy to see that for suitable switching signals  $\sigma(\cdot) \in \Sigma$ , e.g., switching at  $\tau_k = k\delta$ ,  $k \in \mathbb{N}$ , for a fixed  $\delta < 1$ ,

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \rightarrow +\infty \quad \text{as } t \rightarrow +\infty.$$

In fact, the speed of blow-up is not uniformly exponentially bounded over the set of all possible  $\sigma(\cdot)$ , i.e., for any fixed  $t > 0$ , we have

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \geq 2^{\lceil \frac{t}{\delta} \rceil} \rightarrow +\infty \quad \text{as } \delta \rightarrow 0,$$

with  $\lceil \tau \rceil = \min\{k \in \mathbb{N} : \tau \leq k\}$  for  $\tau > 0$  (see Figure 1). This can be seen by taking  $L^1(-1, 1)$ -functions  $f$  of norm one, identically constant near  $x = 0$  on progressively smaller intervals and zero elsewhere.  $\square$

### 3. FIRST CONVERSE LYAPUNOV THEOREM

In this section we establish a first equivalence result for the global uniform exponential stability of a switched of the form (9). The crucial step is given by the following lemma, related to the blow-up phenomenon illustrated in Example 1 in the previous section. It is a variant of a result obtained in [23] in the framework of strongly continuous semigroups. While extending the property to switched system of the form (9), the proof given in [23] should be modified in order to replace the semigroup property by (11). We include the modified proof for the sake of completeness.

**Lemma 2.** *Assume that*

i) there exist constants  $M \geq 1$  and  $\omega > 0$  such that

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}, \quad t \geq 0, \quad \sigma(\cdot)\text{-uniformly};$$

ii) there exists a constant  $C > 0$  and some  $p \in [1, \infty)$  such that

$$\int_0^\infty \|T_{\sigma(\cdot)}(t)x\|_X^p dt \leq C\|x\|_X^p, \quad x \in X, \quad \sigma(\cdot)\text{-uniformly}.$$

Then, there exist constants  $K \geq 1$  and  $\mu > 0$  such that

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Ke^{-\mu t}, \quad t \geq 0, \quad \sigma(\cdot)\text{-uniformly}.$$

*Proof.* First, we show that under the assumptions i) and ii), for every  $x \in X$ , there exists a constant  $C_x > 0$  such that

$$\|T_{\sigma(\cdot)}(t)x\|_X \leq C_x, \quad t \geq 0, \quad \sigma(\cdot)\text{-uniformly} \quad (16)$$

and that, for all  $\sigma(\cdot)$  and for all  $x \in X$ ,

$$\lim_{t \rightarrow +\infty} \|T_{\sigma(\cdot)}(t)x\|_X = 0. \quad (17)$$

To this end, let  $t > \frac{1}{\omega}$  and set  $\Delta(t) = [t - \frac{1}{\omega}, t]$ . Observe that, for every  $\sigma(\cdot)$  and every  $\tau \in \Delta(t)$ , there exists a  $\sigma_\tau(\cdot)$  such that

$$\begin{aligned} \|T_{\sigma(\cdot)}(t)x\|_X &= \|T_{\sigma_\tau(\cdot)}(t - \tau)T_{\sigma(\cdot)}(\tau)x\|_X \\ &\leq \|T_{\sigma_\tau(\cdot)}(t - \tau)\|_{\mathcal{L}(X)} \|T_{\sigma(\cdot)}(\tau)x\|_X. \end{aligned} \quad (18)$$

Moreover, by assumption i) and by definition of  $\Delta(t)$ , we have

$$\|T_{\sigma_\tau(\cdot)}(t - \tau)\|_{\mathcal{L}(X)} \leq Me^{\omega(t - \tau)} \leq Me^{\omega \frac{1}{\omega}} = Me, \quad (19)$$

yielding

$$\|T_{\sigma(\cdot)}(\tau)x\|_X \geq \frac{\|T_{\sigma(\cdot)}(t)x\|_X}{Me}, \quad \tau \in \Delta(t). \quad (20)$$

Now suppose (16) does not hold. Then, there exist  $x \in X$ , a sequence of switching signals  $(\sigma_i(\cdot))_{i \in \mathbb{N}}$  in  $\Sigma$  and a sequence of times  $(t_i)_{i \in \mathbb{N}}$  such that

$$\delta_i = \|T_{\sigma_i(\cdot)}(t_i)x\|_X \rightarrow +\infty \quad \text{as } i \rightarrow +\infty. \quad (21)$$

Assumption i) guarantees that  $t_i$  is diverging. Without loss of generality,  $t_i > \frac{1}{\omega}$  for every  $i \in \mathbb{N}$ .

For  $\tau \in \Delta(t_i)$ , (20) yields

$$\|T_{\sigma_i(\cdot)}(\tau)x\|_X \geq \frac{\delta_i}{Me}, \quad \tau \in \Delta(t_i). \quad (22)$$

Hence, using (22) and again the size of  $\Delta(t_i)$ , we obtain from (21)

$$\begin{aligned} \int_0^\infty \|T_{\sigma_i(\cdot)}(\tau)x\|_X^p d\tau &\geq \int_{\Delta(t_i)} \|T_{\sigma_i(\cdot)}(\tau)x\|_X^p d\tau \\ &\geq \left(\frac{\delta_i}{Me}\right)^p \frac{1}{\omega} \rightarrow \infty \end{aligned}$$

as  $i \rightarrow \infty$ . This contradicts assumption ii). Hence, (16) holds true.

Next, suppose (17) does not hold. Then, there exist  $x \in X$ ,  $\sigma(\cdot) \in \Sigma$ ,  $\delta > 0$  and a diverging sequence of times  $(t_i)_{i \in \mathbb{N}}$  such that

$$\|T_{\sigma(\cdot)}(t_i)x\|_X \geq \delta \quad \text{for all } i. \quad (23)$$

Without loss of generality  $t_i > t_{i-1} + \frac{1}{\omega}$  for every  $i \in \mathbb{N}$  with  $t_0 = 0$ . For  $\tau \in \Delta(t_i)$ , (20) yields

$$\|T_{\sigma(\cdot)}(\tau)x\|_X \geq \frac{\delta}{Me}, \quad \tau \in \Delta(t_i). \quad (24)$$

Hence, using (22) and again the size of  $\Delta(t_i)$ , we obtain

$$\begin{aligned} \int_0^\infty \|T_{\sigma(\cdot)}(\tau)x\|_X^p d\tau &\geq \sum_{i=1}^\infty \int_{\Delta(t_i)} \|T_{\sigma(\cdot)}(\tau)x\|_X^p d\tau \\ &\geq \left(\frac{\delta}{Me}\right)^p \sum_{i=1}^\infty \frac{1}{\omega} = \infty. \end{aligned}$$

This again contradicts assumption ii) and hence (17) holds true.

Let

$$t_{x,\sigma(\cdot)}(\rho) = \max\{t : \|T_{\sigma(\cdot)}(t)x\|_X \geq \rho\|x\|_X, 0 \leq t \leq t\}.$$

By (17),  $t_{x,\sigma(\cdot)}(\rho)$  is finite (and positive) for every  $\sigma(\cdot)$  and  $x \in X \setminus \{0\}$ . By strong continuity,

$$\|T_{\sigma(\cdot)}(t_{x,\sigma(\cdot)}(\rho))x\|_X = \rho\|x\|_X.$$

Using assumption ii),

$$\begin{aligned} t_{x,\sigma(\cdot)}(\rho)\rho^p\|x\|_X^p &\leq \int_0^{t_{x,\sigma(\cdot)}(\rho)} \|T_{\sigma(\cdot)}(t)x\|_X^p dt \\ &\leq \int_0^\infty \|T_{\sigma(\cdot)}(t)x\|_X^p dt \leq C\|x\|_X^p \end{aligned}$$

whereby,

$$t_{x,\sigma(\cdot)}(\rho) \leq \frac{C}{\rho^p} =: t_0, \text{ independent of } \sigma(\cdot).$$

By the principle of uniform boundedness, (16) implies that there exists a constant  $k > 0$  such that

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq k, \quad t \geq 0, \sigma(\cdot)\text{-uniformly.} \quad (25)$$

Hence, for  $t > t_0$ , we have

$$\begin{aligned} \|T_{\sigma(\cdot)}(t)x\|_X &\leq \sup_{\tilde{\sigma}(\cdot)} \|T_{\tilde{\sigma}(\cdot)}(t - t_{x,\sigma(\cdot)}(\rho))\|_{\mathcal{L}(X)} \rho\|x\|_X \\ &\leq k\rho\|x\|_X, \quad \sigma(\cdot)\text{-uniformly.} \end{aligned}$$

Choose  $\rho > 0$  such that  $\beta := k\rho < 1$ , so that

$$\|T_{\sigma(\cdot)}(t)x\|_X \leq \beta\|x\|_X, \quad t \geq t_0, \sigma(\cdot)\text{-uniformly.}$$

Finally, let  $t_1 > t_0$  be fixed and let  $t = nt_1 + s$ ,  $0 \leq s < t_1$ . Then,

$$\begin{aligned} \|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} &\leq \sup_{\tilde{\sigma}(\cdot), \hat{\sigma}(\cdot)} \|T_{\tilde{\sigma}(\cdot)}(s)\|_{\mathcal{L}(X)} \|T_{\hat{\sigma}(\cdot)}(nt_1)\|_{\mathcal{L}(X)} \\ &\leq k \left( \sup_{\hat{\sigma}(\cdot)} \|T_{\hat{\sigma}(\cdot)}(t_1)\|_{\mathcal{L}(X)} \right)^n \\ &\leq k\beta^n \leq Ke^{-\mu t}, \quad t \geq 0, \sigma(\cdot)\text{-uniformly} \end{aligned}$$

with  $K = \frac{C}{\beta}$  and  $\mu = -\left(\frac{1}{t_1}\right) \ln \beta > 0$ .  $\square$

Lemma 2 allows us to prove the first of the converse Lyapunov theorems that are the object of this paper.

**Theorem 3.** *The following conditions:*

i) *there exist constants  $M \geq 1$  and  $\omega > 0$  such that*

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}, \quad t \geq 0, \sigma(\cdot)\text{-uniformly,} \quad (26)$$



ii) there exists  $V: X \rightarrow [0, \infty)$  such that  $\sqrt{V(\cdot)}$  is a norm on  $X$ ,

$$V(x) \leq C\|x\|_X^2, \quad x \in X, \quad (27)$$

for a constant  $C > 0$  and

$$\underline{L}_j V(x) \leq -\|x\|_X^2, \quad j \in Q, \quad x \in X, \quad (28)$$

with  $\underline{L}_j V(x)$  defined as in (12)

are necessary and sufficient for the existence of constants  $K \geq 1$  and  $\mu > 0$  such that

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq K e^{-\mu t}, \quad t \geq 0, \quad \sigma(\cdot)\text{-uniformly.} \quad (29)$$

*Proof.* Assume that the conditions i) and ii) hold. For all  $\sigma(\cdot) \in \Sigma$ ,  $x \in X$  and for  $t \geq 0$  small enough so that the restriction of  $\sigma(\cdot)$  to the interval  $[0, t]$  is constant, we have

$$0 \leq V(T_{\sigma(\cdot)}(t)x) \leq V(x) - \int_0^t \|T_{\sigma(\cdot)}(\tau)x\|_X^2 d\tau$$

as it follows from (28) and [20, §VI.7] (see also [10]). Thus, for all  $\sigma(\cdot)$  and  $x \in X$ ,

$$\int_0^\infty \|T_{\sigma(\cdot)}(\tau)x\|_X^2 d\tau \leq V(x) \leq C\|x\|_X^2. \quad (30)$$

The uniform exponential decay (29) now follows from (26) and (30), thanks to Lemma 2 with  $p = 2$ .

Conversely, assume that (29) holds for some constants  $K \geq 1$  and  $\mu > 0$ . Then, (26) holds for  $M = K$  and arbitrary  $\omega > 0$ . Define  $V: X \rightarrow [0, \infty)$  by

$$V(x) = \sup_{\sigma(\cdot) \in \Sigma} \int_0^\infty \|T_{\sigma(\cdot)}(t)x\|_X^2 dt. \quad (31)$$

Then, by assumption,  $V(x)$  satisfies

$$V(x) \leq \sup_{\sigma(\cdot)} \int_0^\infty K^2 e^{-2\mu t} \|x\|_X^2 dt = \frac{K^2}{2\mu} \|x\|_X^2,$$

establishing (27) with  $C = \frac{K^2}{2\mu}$ . In particular,  $V$  is well-posed.

Notice that, by definition,  $V$  is positive definite and homogenous of degree 2. In order to show that it is the square of a norm, we are left to prove that it is convex and continuous.

The convexity of  $V$  follows from the fact that each

$$x \mapsto \int_0^\infty \|T_{\sigma(\cdot)}(t)x\|_X^2 dt$$

is convex.

In order to verify the continuity of  $V$ , let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $X$  converging to  $x$  in  $X$ . By definition of  $V$ ,

$$\int_0^\infty \|T_{\sigma(\cdot)}(t)x_n\|_X^2 dt \leq V(x_n), \quad (32)$$

for all  $\sigma(\cdot) \in \Sigma$ . So taking the  $\liminf$  over  $n \in \mathbb{N}$  in (32) on both sides and using the continuity of  $T_{\sigma(\cdot)}(t)$  for all  $t \geq 0$ , we have

$$\int_0^\infty \|T_{\sigma(\cdot)}(t)x\|_X^2 dt \leq \liminf_{n \rightarrow \infty} \int_0^\infty \|T_{\sigma(\cdot)}(t)x_n\|_X^2 dt \leq \liminf_{n \rightarrow \infty} V(x_n). \quad (33)$$

Taking the sup over  $\sigma(\cdot)$  in (33) then yields

$$V(x) = \sup_{\sigma(\cdot)} \int_0^\infty \|T_{\sigma(\cdot)}(t)x\|_X^2 dt \leq \liminf_{n \rightarrow \infty} V(x_n),$$

proving that  $V$  is lower semi-continuous. On the other hand, for a fixed  $\varepsilon > 0$ , there exist  $\sigma_\varepsilon(\cdot) \in \Sigma$  such that

$$\begin{aligned} V(x_n) - \frac{\varepsilon}{2} &< \int_0^\infty \|T_{\sigma_\varepsilon(\cdot)}(t)x_n\|^2 dt \\ &\leq (1+m) \int_0^\infty \|T_{\sigma_\varepsilon(\cdot)}(t)(x_n - x)\|^2 dt + \left(1 + \frac{1}{m}\right) \int_0^\infty \|T_{\sigma_\varepsilon(\cdot)}(t)x\|^2 dt, \end{aligned}$$

for any  $m > 0$ . Notice that, by definition of  $V$ ,

$$\begin{aligned} \int_0^\infty \|T_{\sigma_\varepsilon(\cdot)}(t)(x_n - x)\|^2 dt &\leq V(x_n - x) \leq C\|x_n - x\|_X^2, \\ \int_0^\infty \|T_{\sigma_\varepsilon(\cdot)}(t)x\|^2 dt &\leq V(x). \end{aligned}$$

Thus, for any  $m > 0$ , we have

$$V(x_n) - \frac{\varepsilon}{2} < C(1+m)\|x_n - x\|_X^2 + \left(1 + \frac{1}{m}\right)V(x). \quad (34)$$

In particular, choosing  $m$  such that  $(1 + \frac{1}{m})V(x) < V(x) + \frac{\varepsilon}{4}$  and taking  $n$  sufficiently large, so that  $(1+m)C\|x_n - x\|_X^2 \leq \frac{\varepsilon}{4}$ , we have from (34)

$$V(x_n) < V(x) + \varepsilon, \quad n \text{ sufficiently large.}$$

This implies the upper semi-continuity of  $V$ . Resuming, we proved the continuity of  $V$ .

To complete the proof of the theorem, we are left to show that  $V$  satisfies (28). Fixing  $t > 0$ ,  $j \in Q$  and letting

$$\Sigma_{t,j} = \{\sigma(\cdot) \in \Sigma : \sigma|_{[0,t]} \equiv j\}$$

be the set of switching signals whose restriction to the interval  $[0, t]$  is constantly equal to  $j$ , we have, since  $\Sigma_{t,j} \subseteq \Sigma$ ,

$$\begin{aligned} V(x) &\geq \sup_{\sigma(\cdot) \in \Sigma_{t,j}} \int_0^\infty \|T_{\sigma(\cdot)}(\tau)x\|_X^2 d\tau \\ &= \int_0^t \|T_j(\tau)x\|_X^2 d\tau + \sup_{\sigma(\cdot) \in \Sigma_{t,j}} \int_t^\infty \|T_{\sigma(\cdot)}(\tau)x\|_X^2 d\tau. \end{aligned} \quad (35)$$

Moreover, thanks to (11) and the invariance of  $\Sigma$  by time-shift,

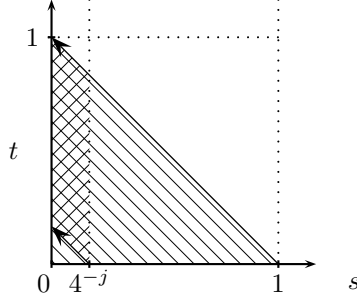
$$\begin{aligned} V(T_j(t)x) &= \sup_{\sigma(\cdot) \in \Sigma} \int_0^\infty \|T_{\sigma(\cdot)}(\tau)T_j(t)x\|_X^2 d\tau \\ &= \sup_{\sigma(\cdot) \in \Sigma} \int_t^\infty \|T_{\sigma(\cdot)}(\tau - t)T_j(t)x\|_X^2 d\tau \\ &= \sup_{\sigma(\cdot) \in \Sigma_{t,j}} \int_t^\infty \|T_{\sigma(\cdot)}(\tau)x\|_X^2 d\tau. \end{aligned}$$

This and (35) yield

$$V(T_j(t)x) - V(x) \leq - \int_0^t \|T_j(\tau)x\|_X^2 d\tau,$$

for all  $j \in Q$  and  $t > 0$ . Therefore

$$\begin{aligned} L_j V(x) &= \liminf_{t \downarrow 0} \frac{V(T_j(t)x) - V(x)}{t} \\ &\leq - \limsup_{t \downarrow 0} \frac{1}{t} \int_0^t \|T_j(\tau)x\|_X^2 d\tau = -\|x\|_X^2 \end{aligned}$$

FIGURE 2. Illustration of  $T_j(t)$ ,  $j \in Q$ , of the example in Remark 4.

for all  $j \in Q$ , establishing (28).  $\square$

*Remark 4.* We show here, through an example, that condition i) appearing in the statement of Theorem 3 cannot be removed.

Consider the family of semigroups  $\{T_j(t)\}_{j \in Q}$  with  $Q = \mathbb{N}$  and  $T_j(t)$  defined on the Lebesgue space  $X = L^p(0, 1)$ ,  $p \in [1, \infty)$ , by

$$(T_j(t)f)(s) = \begin{cases} 2^{\frac{1}{p}} f(s+t), & s \in [0, 1-t] \cap [0, 4^{-j}) \\ f(s+t), & s \in [0, 1-t] \setminus [0, 4^{-j}) \\ 0, & s \in (1-t, 1], \end{cases} \quad (36)$$

cf. also Figure 2. Notice that, for all  $j \in Q$ ,  $T_j(\cdot)$  is a nilpotent semigroup, since  $\|T_j(t)f\|_X = 0$  for  $t > 1$ . In particular, each of them is exponentially stable. Moreover, for all  $\sigma(\cdot) \in \Sigma$ , we have

$$|(T_{\sigma(\cdot)}(t)f)(s)| \leq 2^{\frac{k}{p}} |f(s+t)|, \quad \text{with } k = \#\{l \in \mathbb{N} : s < 4^{-l} \leq s+t\}.$$

In particular,

$$|(T_{\sigma(\cdot)}(t)f)(s)| \leq 2^{\frac{k}{p}} |f(s+t)|, \quad \text{if } 4^{-k-1} \leq s < 4^{-k},$$

yielding

$$\begin{aligned} \int_0^\infty \|T_{\sigma(\cdot)}(t)f\|_X^p dt &= \int_0^1 \int_0^{1-t} |(T_{\sigma(\cdot)}(t)f)(s)|^p ds dt \\ &= \int_0^1 \int_0^{1-s} |(T_{\sigma(\cdot)}(t)f)(s)|^p dt ds \\ &\leq \sum_{k=0}^\infty \int_{4^{-k-1}}^{4^{-k}} 2^k \left( \int_0^{1-s} |f(s+t)|^p dt \right) ds \\ &\leq \sum_{k=0}^\infty \left( \frac{1}{4^k} - \frac{1}{4^{k+1}} \right) 2^k \int_0^1 |f(t)|^p dt = \frac{3}{2} \|f\|_X^p. \end{aligned}$$

Hence, defining  $V(x)$  as in (31), we have

$$V(x) = \sup_{\sigma(\cdot)} \int_0^\infty \|T_{\sigma(\cdot)}(t)x\|_X^2 dt \leq \frac{3}{2} \|x\|_X^2$$

and, by the same arguments as in the proof of Theorem 3,

$$L_j V(x) \leq -\|x\|_X^2, \quad \text{for all } j \in Q,$$

so condition ii) in Theorem 3 holds with  $C = \frac{3}{2}$ .

Nevertheless, for a sequence of switching signals  $(\sigma_n(\cdot))_{n \in \mathbb{N}} \subset \Sigma$  with switching times  $\tau_k = \frac{1}{4^k}$  and modes  $j_k = k + 1$ ,  $0 \leq k \leq n$ , we have for functions  $\mathbb{1}_{[s,1]}$  of  $L^p$  norm one concentrated on the interval  $[s, 1]$ ,

$$T_{\sigma_n(\cdot)}(1 - \epsilon) \mathbb{1}_{[s,1]} = 2^{\frac{n}{p}} \mathbb{1}_{[s-1+\epsilon, \epsilon]}, \quad \text{if } 1 \geq s \geq 1 - \epsilon > 1 - 4^{-n}.$$

Therefore, for  $\epsilon < 4^{-n}$ ,

$$\begin{aligned} \|T_{\sigma_n(\cdot)}(1 - \epsilon)\|_{\mathcal{L}(X)} &= \sup_{\|f\|_X=1} \|T_{\sigma_n(\cdot)}(1 - \epsilon)f\|_X \\ &\geq \lim_{s \uparrow 1} \|T_{\sigma_n(\cdot)}(1 - \epsilon) \mathbb{1}_{[s,1]}\|_X = 2^{\frac{n}{p}}. \end{aligned}$$

Hence,

$$\sup\{\|T_{\sigma_n(\cdot)}(1 - \epsilon)\|_{\mathcal{L}(X)} : \epsilon \in [0, 1], n \in \mathbb{N}\} = +\infty \quad (37)$$

violating any uniform bound of the form (29).

This example also shows that assumption i) appearing in Lemma 2 is necessary for the validity of the lemma.  $\square$

#### 4. SECOND CONVERSE LYAPUNOV THEOREM

If one wishes to conclude that a switched system is globally uniformly exponentially stable, Theorem 3 requires the knowledge of a squared norm  $V(\cdot)$  satisfying (27), (28) for all modes  $j \in Q$  and the knowledge of a global uniform exponential bound (26). As an alternative, we will, in Theorem 6, show that the existence of a Lyapunov norm  $\sqrt{V(\cdot)}$  that is comparable with the norm  $\|\cdot\|_X$  allows to conclude that the system is globally uniformly exponentially stable, without knowledge of a global uniform exponential bound of the type (26). Notice that the norm  $\sqrt{V(\cdot)}$  constructed in the proof of Theorem 3 (see definition (31)) is in general not comparable with  $\|\cdot\|_X$ , i. e., in general it does not satisfy a lower bound of the form

$$c\|x\|_X \leq \sqrt{V(x)}, \quad x \in X \quad (38)$$

for a constant  $c > 0$ . Such a lower bound always holds, on the contrary, when  $X$  has finite dimension, as it is exploited in [17, 8]. The bound (38) may fail to hold even in the case of a single strongly continuous semigroup, as it is the case, for instance, of the semigroups  $T_1(\cdot)$  and  $T_2(\cdot)$  introduced in Example 1. (For a characterization of exponentially stable strongly continuous semigroups whose Lyapunov function defined as in (31) is comparable with the squared norm, see [19].)

In order to obtain a Lyapunov norm comparable with  $\|\cdot\|_X$  for infinite dimensional switched systems, we make use of the following lemma imposing a stronger assumption on the family of semigroups  $T_j(\cdot)$ .

**Lemma 5.** *Assume that there exists  $j^* \in Q$  such that  $T_{j^*}(\cdot)$  can be extended to a group of bounded linear operators on  $X$ . Moreover, assume that there exist constants  $K \geq 1$  and  $\mu > 0$  such that*

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq K e^{-\mu t}, \quad t \geq 0, \sigma(\cdot)\text{-uniformly}. \quad (39)$$

*Then there exists a function  $V: X \rightarrow [0, \infty)$  such that  $\sqrt{V(\cdot)}$  is a norm on  $X$ ,*

$$c\|x\|_X^2 \leq V(x) \leq C\|x\|_X^2, \quad x \in X, \quad (40)$$

*for constants  $c, C > 0$  and*

$$L_j V(x) \leq -\|x\|_X^2, \quad j \in Q, x \in X, \quad (41)$$

*with  $L_j V(x)$  defined as in (12).*

*Proof.* Assume that (39) holds for some constants  $K \geq 1$  and  $\mu > 0$  independent of  $\sigma(\cdot)$ . Define  $V(\cdot)$  by (31). As seen in the proof of Theorem 3, (39) guarantees that  $V$  is the square of a norm, satisfies (41) and that there exist  $C > 0$  such that  $V(x) \leq C\|x\|_X^2$ .

The remaining bound  $c\|x\|^2 \leq V(x)$  for some constant  $c > 0$  is a consequence of the assumption that  $T_{j^*}(\cdot)$  can be extended to a group. Indeed,  $T_{j^*}(t)$  is then invertible for every  $t \geq 0$  and satisfies  $\|T_{j^*}(t)x\|_X \geq (\|T_{j^*}(-t)\|_{\mathcal{L}(X)})^{-1}\|x\|_X$  (cf. [19]). Hence

$$\begin{aligned} V(x) &= \sup_{\sigma(\cdot)} \int_0^\infty \|T_{\sigma(\cdot)}(t)x\|_X^2 dt \geq \int_0^\infty \|T_{j^*}(t)x\|_X^2 dt \\ &\geq \int_0^\infty (\|T_{j^*}(-t)\|_{\mathcal{L}(X)})^{-2} dt \|x\|_X^2 = c\|x\|_X^2 \end{aligned}$$

with  $c = \int_0^\infty (\|T_{j^*}(-t)\|_{\mathcal{L}(X)})^{-2} dt < \infty$ .  $\square$

We can now state and prove our second converse Lyapunov theorem.

**Theorem 6.** *The existence of a function  $V: X \rightarrow [0, \infty)$  such that  $\sqrt{V(\cdot)}$  is a norm on  $X$ ,*

$$c\|x\|_X^2 \leq V(x) \leq C\|x\|_X^2, \quad x \in X, \quad (42)$$

*for constants  $c, C > 0$  and*

$$\underline{L}_j V(x) \leq -\|x\|_X^2, \quad j \in Q, \quad x \in X, \quad (43)$$

*with  $\underline{L}_j V(x)$  defined as in (12) is necessary and sufficient for the existence of constants  $K \geq 1$  and  $\mu > 0$  such that*

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq K e^{-\mu t}, \quad t \geq 0, \quad \sigma(\cdot)\text{-uniformly}. \quad (44)$$

*Proof.* Assume that there exists a function  $V: X \rightarrow [0, \infty)$  such that (42) and (43) hold for  $c, C > 0$ . Then, for all  $x \in X$ ,  $\sigma(\cdot) \in \Sigma$ , and  $t \geq 0$ ,

$$c\|T_{\sigma(\cdot)}(t)x\|_X^2 \leq V(T_{\sigma(\cdot)}(t)x) \leq V(x) - \int_0^t \|T_{\sigma(\cdot)}(\tau)x\|_X^2 d\tau,$$

so, using again (42) and dividing by  $c$ ,

$$\|T_{\sigma(\cdot)}(t)x\|_X^2 \leq \frac{C}{c}\|x\|_X^2 - \frac{1}{c} \int_0^t \|T_{\sigma(\cdot)}(\tau)x\|_X^2 d\tau.$$

From Gronwall's Lemma, we obtain

$$\|T_{\sigma(\cdot)}(t)x\|_X^2 \leq \frac{C}{c} e^{-\frac{1}{c}t} \|x\|_X^2, \quad t \geq 0, \quad \sigma(\cdot)\text{-uniformly}.$$

Hence,

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq K e^{-\mu t}, \quad t \geq 0, \quad \sigma(\cdot)\text{-uniformly},$$

for the constants  $K = \sqrt{\frac{C}{c}} \geq 1$  and  $\mu = \frac{1}{2c} > 0$ , establishing (44).

Conversely, assume that (44) holds for  $K \geq 1$  and  $\mu > 0$  and consider the switched system with  $\sigma^*(\cdot)$  taking values in  $Q^* = Q \cup \{j^*\}$ , where

$$T_{j^*}(t) = e^{-\mu t} I \quad (45)$$

and  $I: X \rightarrow X$  denotes the identity. Then,

$$T_{j^*}(t)T_j(s) = T_j(s)T_{j^*}(t), \quad t, s \geq 0, \quad j \in Q.$$

Moreover, for  $t^* = |\{\tau \in [0, t] : \sigma^*(\tau) = j^*\}|$  and some  $\sigma(\cdot)$  just taking values in  $Q$ ,

$$\begin{aligned} \|T_{\sigma^*(\cdot)}\|_{\mathcal{L}(X)} &= \|T_{\sigma(\cdot)}(t - t^*)e^{-\mu t^*}\|_{\mathcal{L}(X)} \leq Ke^{-\mu(t-t^*)}e^{-\mu t^*} \\ &= Ke^{-\mu t}, \quad t \geq 0, \sigma(\cdot)\text{-uniformly} \end{aligned} \quad (46)$$

where we have used (44). The existence of a squared norm  $V: X \rightarrow [0, \infty)$  and constants  $c, C > 0$  such that (42) and (43) hold now follows from (46) and Lemma 5, noting that (45) actually defines a group.  $\square$

*Remark 7.* In the case of a single strongly continuous semigroup  $T(\cdot)$ , Theorem 6 shows that its global exponential stability is equivalent to the existence of a Lyapunov norm comparable with the norm  $\|\cdot\|_X$ .

When  $T(\cdot)$  is globally exponentially stable, such a Lyapunov norm can be obtained by the construction suggested in the proof of Theorem 6.

An alternative construction of a common Lyapunov function has been proposed in [12], where (31) is replaced by

$$\tilde{V}(x) = \int_0^\infty \sup_{\sigma(\cdot) \in \Sigma} \|T_{\sigma(\cdot)}(t)x\|_X^2 dt. \quad (47)$$

In the case of a single mode and following the strategy of augmenting  $Q$  by adding a group corresponding to a diagonal operator, this construction leads to the explicit expression

$$\tilde{V}(x) = \int_0^\infty \left( \max_{s \in [0, \tau]} e^{2\mu(s-\tau)} \|T(s)x\|_X^2 \right) d\tau,$$

for any fixed  $\mu > 0$ .

It should be noticed that, although not stated in [12], the definition of  $\tilde{V}$  given in (47) identifies a function which is positive definite, homogenous of degree 2, continuous and convex, i.e., a squared norm. The proof of this fact can be rather easily obtained by adapting the proof of Theorem 3.  $\square$

## 5. COMMON LYAPUNOV FUNCTIONS ON HILBERT SPACES

The goal of this section is to prove further regularity properties of the common Lyapunov functions constructed in the previous sections in the case in which  $X$  is a separable Hilbert space. The proof of the following lemma adapts arguments presented in [4, §4.3.1].

Recall that a function  $V: X \rightarrow \mathbb{R}$  is said to be *directionally differentiable in the sense of Fréchet* at  $x \in X$  if for every  $\psi \in X$  there exists

$$V'(x, \psi) = \lim_{t \downarrow 0} \frac{V(x + t\psi) - V(x)}{t}$$

and, moreover,

$$\lim_{\psi \rightarrow 0} \frac{V(x + \psi) - V(x) - V'(x, \psi)}{\|\psi\|_X} = 0.$$

Notice that, in general,  $V'(x, \cdot)$  needs not be linear.

**Lemma 8.** *Let  $X$  be a separable Hilbert space with scalar product  $\langle \cdot, \cdot \rangle$  and assume that there exist constants  $K \geq 1$  and  $\mu > 0$  such that*

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Ke^{-\mu t}, \quad t \geq 0, \sigma(\cdot)\text{-uniformly}. \quad (48)$$

*Then, there exists a subset  $\mathcal{B}$  of  $\mathcal{L}(X)$ , compact for the weak operator topology and made of self-adjoint operators, such that*

$$\sup_{\sigma(\cdot) \in \Sigma} \int_0^\infty \|T_{\sigma(\cdot)}(t)x\|_X^2 dt = \max_{B \in \mathcal{B}} \langle x, Bx \rangle =: V(x). \quad (49)$$

In particular,  $V$  is directionally differentiable in the sense of Fréchet and its derivative in the direction  $\psi \in X$  is given by

$$V'(x, \psi) = \max_{\hat{B} \in \mathcal{S}(x)} 2\langle \psi, \hat{B}x \rangle, \quad (50)$$

where

$$\mathcal{S}(x) = \arg \max_{B \in \mathcal{B}} \langle x, Bx \rangle. \quad (51)$$

*Proof.* For a fixed  $\sigma(\cdot) \in \Sigma$  and all  $t \geq 0$ , let  $T_{\sigma(\cdot)}^*(t) \in \mathcal{L}(X)$  be the adjoint operator of  $T_{\sigma(\cdot)}(t) \in \mathcal{L}(X)$ , uniquely defined by

$$\langle T_{\sigma(\cdot)}(t)x, x' \rangle = \langle x, T_{\sigma(\cdot)}^*(t)x' \rangle, \quad x, x' \in X.$$

If  $T_{\sigma(\cdot)}(t)$  has the expression given in (10), then

$$T_{\sigma(\cdot)}^*(t) = T_{j_1}^*(\tau_1) \cdots T_{j_{p-1}}^*(\tau_p - \tau_{p-1}) T_{j_p}^*(t - \tau_p),$$

where  $T_j^*(t)$  denotes the adjoint semigroup of  $T_j(t)$ ,  $t \geq 0$ ,  $j \in Q$ .

Assuming that there exist constants  $K \geq 1$  and  $\mu > 0$ , independent of  $\sigma(\cdot)$ , such that (48) holds, the operator  $B_{\sigma(\cdot)}: X \rightarrow X$ , given by

$$B_{\sigma(\cdot)}x = \int_0^\infty T_{\sigma(\cdot)}^*(t) T_{\sigma(\cdot)}(t)x \, dt,$$

is linear, self-adjoint, and satisfies

$$\begin{aligned} \|B_{\sigma(\cdot)}x\|_X &\leq \int_0^\infty \|T_{\sigma(\cdot)}^*(t)\|_{\mathcal{L}(X)} \|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \|x\|_X \, dt \\ &\leq \int_0^\infty K^2 e^{-2\mu t} \|x\|_X \, dt = \frac{K^2}{2\mu} \|x\|_X, \end{aligned}$$

where we have used (48). In particular,  $B_{\sigma(\cdot)} \in \mathcal{L}(X)$  for all  $\sigma(\cdot) \in \Sigma$  and

$$\|B_{\sigma(\cdot)}\|_{\mathcal{L}(X)} \leq \frac{K^2}{2\mu}, \quad \sigma(\cdot)\text{-uniformly}. \quad (52)$$

Therefore, the set

$$\mathcal{B} = \{B \in \mathcal{L}(X) : \text{there exists a sequence } (\sigma_n(\cdot))_{n \in \mathbb{N}} \subset \Sigma \text{ such that } B_{\sigma_n(\cdot)} \xrightarrow{\text{WOT}} B\}$$

is compact for the weak operator topology. We recall that  $B_{\sigma_n(\cdot)} \xrightarrow{\text{WOT}} B$  (i.e.,  $B_{\sigma_n(\cdot)}$  converges to  $B$  for the *weak operator topology*) if, for every sequence  $(x_n, y_n)$  converging to  $(x, y)$  in  $X \times X$ , we have

$$\lim_{n \rightarrow \infty} \langle x_n, B_{\sigma_n(\cdot)} y_n \rangle = \langle x, By \rangle, \quad (53)$$

and that every bounded closed subset of  $\mathcal{L}(X)$  is sequentially compact for the weak operator topology (see, for instance, [9, Theorem III.4]). Notice that (53) guarantees that  $\mathcal{B}$  consists of self-adjoint operators.

Define  $V$  as in (49). The maximization makes sense because of (53) (with  $x_n = y_n = x$ ) and of the compactness of  $\mathcal{B}$ . Moreover,

$$\begin{aligned} V(x) &= \max_{B \in \mathcal{B}} \langle x, Bx \rangle = \sup_{\sigma(\cdot) \in \Sigma} \langle x, B_{\sigma(\cdot)}x \rangle = \sup_{\sigma(\cdot) \in \Sigma} \langle x, \int_0^\infty T_{\sigma(\cdot)}^*(t) T_{\sigma(\cdot)}(t)x \, dt \rangle \\ &= \sup_{\sigma(\cdot) \in \Sigma} \int_0^\infty \langle x, T_{\sigma(\cdot)}^*(t) T_{\sigma(\cdot)}(t)x \rangle \, dt = \sup_{\sigma(\cdot) \in \Sigma} \int_0^\infty \|T_{\sigma(\cdot)}(t)x\|^2 \, dt, \end{aligned}$$

for all  $x \in X$ . Hence  $V$  coincides with the Lyapunov function defined in (31) and we recover that  $V$  satisfies the condition ii) of Theorem 3. In particular,  $V$  is continuous.

In order to verify the directional differentiability in the sense of Fréchet and to prove (50), we show below that

$$\lim_{\psi \rightarrow 0} \frac{V(x + \psi) - V(x) - \max_{\hat{B} \in \mathcal{S}(x)} 2\langle \psi, \hat{B}x \rangle}{\|\psi\|_X} = 0. \quad (54)$$

First observe that the map  $x \mapsto \langle x, Bx \rangle$  is differentiable on  $X$  in the sense of Fréchet for all  $B \in \mathcal{L}(X)$ . For  $B$  self-adjoint, the derivative is given by  $2\langle \cdot, Bx \rangle$ . Fix now some  $x_0 \in X$  and define

$$\Phi(B, x) = 2\|Bx - Bx_0\|_X.$$

We claim that

$$\lim_{x \rightarrow x_0} \Phi(B, x) = 0, \quad \text{uniformly with respect to } B \in \mathcal{B}. \quad (55)$$

Indeed, suppose by contradiction that there exists some  $\varepsilon > 0$ , a sequence  $(x_n)_{n \in \mathbb{N}}$  converging to  $x_0$  in  $X$  and a sequence  $(B_n)_{n \in \mathbb{N}}$  in  $\mathcal{B}$  such that

$$\Phi(B_n, x_n) > \varepsilon \quad \text{for all } n \in \mathbb{N}. \quad (56)$$

Thanks to the compactness of  $\mathcal{B}$ , there exist  $B \in \mathcal{B}$  and a subsequence  $n(k)$  such that

$$B_{n(k)} \xrightarrow{\text{WOT}} B \quad \text{as } k \rightarrow \infty.$$

Then, it follows from (53) that

$$\Phi(B_{n(k)}, x_{n(k)}) = 0,$$

contradicting (56).

The Mean Value Theorem then gives

$$\begin{aligned} & |\langle x_0 + \psi, B(x_0 + \psi) \rangle - \langle x_0, Bx_0 \rangle - 2\langle \psi, Bx_0 \rangle| \\ & \leq \|\psi\|_X \int_0^1 \Phi(B, x_0 + \xi\psi) d\xi \leq \varepsilon \|\psi\|_X \end{aligned} \quad (57)$$

for  $\varepsilon > 0$ ,  $\psi$  sufficiently close to zero, and uniformly with respect to  $B \in \mathcal{B}$ , as it follows from (55).

Let  $\mathcal{S}(\cdot)$  be defined as in (51). Notice that, for every  $x \in X$ ,  $\mathcal{S}(x)$  is close and therefore compact for the weak operator topology. For any  $\hat{B} \in \mathcal{S}(x_0)$ ,  $V(x_0) = \langle x_0, \hat{B}x_0 \rangle$  and  $V(x_0 + \psi) \geq \langle x_0 + \psi, \hat{B}(x_0 + \psi) \rangle$ . Thus, using (57),

$$V(x_0 + \psi) - V(x_0) \geq \max_{\hat{B} \in \mathcal{S}(x_0)} 2\langle \psi, \hat{B}x_0 \rangle - \varepsilon \|\psi\|_X.$$

In order to prove (54), it therefore remains to show that for all  $\varepsilon > 0$  and  $\psi$  close to zero,

$$V(x_0 + \psi) - V(x_0) \leq \max_{\hat{B} \in \mathcal{S}(x_0)} 2\langle \psi, \hat{B}x_0 \rangle + \varepsilon \|\psi\|_X. \quad (58)$$

So, suppose (58) not to hold. Then there exist  $\varepsilon > 0$  and a sequence  $(\psi_n)_{n \in \mathbb{N}}$  in  $X$  converging to zero such that

$$V(x_0 + \psi_n) - V(x_0) > \max_{\hat{B} \in \mathcal{S}(x_0)} 2\langle \psi_n, \hat{B}x_0 \rangle + \varepsilon \|\psi_n\|_X. \quad (59)$$

By definition of  $V$ , there exist  $B_0, \hat{B}_n \in \mathcal{B}$  such that

$$\begin{aligned} V(x_0 + \psi_n) - V(x_0) &= \langle x_0 + \psi_n, \hat{B}_n(x_0 + \psi_n) \rangle - \langle x_0, B_0x_0 \rangle \\ &\leq \langle x_0 + \psi_n, \hat{B}_n(x_0 + \psi_n) \rangle - \langle x_0, \hat{B}_n x_0 \rangle. \end{aligned}$$



Again by the Mean Value Theorem,

$$\begin{aligned} & |\langle x_0 + \psi_n, \hat{B}_n(x_0 + \psi_n) \rangle - \langle x_0, \hat{B}_n x_0 \rangle - 2\langle \psi_n, \hat{B}_n x_0 \rangle| \\ & \leq \|\psi_n\|_X \int_0^1 \Phi(\hat{B}_n, x_0 + \psi_n) dt \leq \frac{\varepsilon}{2} \|\psi_n\|_X \end{aligned}$$

for all  $n$  large enough. Thus,

$$V(x_0 + \psi_n) - V(x_0) \leq 2\langle \psi_n, \hat{B}_n x_0 \rangle + \frac{\varepsilon}{2} \|\psi_n\|_X.$$

Using one more time the compactness of  $\mathcal{B}$ , there exist a subsequence  $n(k)$  such that  $\hat{B}_{n(k)} \xrightarrow{\text{WOT}} \hat{B}$  for some  $\hat{B} \in \mathcal{B}$ . Moreover, by continuity of  $V$  and because of (53),  $\hat{B} \in \mathcal{S}(x_0)$ . Hence,

$$\begin{aligned} V(x_0 + \psi_{n(k)}) - V(x_0) & \leq 2\langle \psi_{n(k)}, \hat{B} x_0 \rangle + 2\langle \psi_{n(k)}, (\hat{B}_{n(k)} - \hat{B}) x_0 \rangle + \frac{\varepsilon}{2} \|\psi_{n(k)}\|_X \\ & \leq \max_{B \in \mathcal{S}(x_0)} 2\langle \psi_{n(k)}, B x_0 \rangle + \frac{3}{4} \varepsilon \|\psi_{n(k)}\|_X \end{aligned}$$

where we used that  $2\langle \psi_{n(k)}, (\hat{B}_{n(k)} - \hat{B}) x_0 \rangle \leq \frac{\varepsilon}{4} \|\psi_{n(k)}\|_X$  for  $k$  sufficiently large. This contradicts (59) and completes the proof.  $\square$

The following Corollary is now a direct consequence of Lemma 8 and the choice of the Lyapunov function  $V(\cdot)$  in the proof of Theorem 3.

**Corollary 9.** *Let  $X$  be a separable Hilbert space and assume that there exist constants  $K \geq 1$  and  $\mu > 0$  such that*

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq K e^{-\mu t}, \quad t \geq 0, \quad \sigma(\cdot)\text{-uniformly.} \quad (60)$$

*Then, there exists a function  $V: X \rightarrow [0, \infty)$  such that  $\sqrt{V(\cdot)}$  is a norm on  $X$ ,  $V(\cdot)$  is directionally Fréchet differentiable,*

$$c\|x\|_X^2 \leq V(x) \leq C\|x\|_X^2, \quad x \in X, \quad (61)$$

*for constants  $c, C > 0$  and*

$$\underline{L}_j V(x) \leq -\|x\|_X^2, \quad j \in Q, \quad x \in X, \quad (62)$$

*with  $\underline{L}_j V(x)$  defined as in (12).*

## 6. FINAL REMARKS AND OPEN PROBLEMS

We presented necessary and sufficient conditions for a (possibly infinite) family of semigroups to be globally uniformly exponentially stable for arbitrary switching signals  $\sigma(\cdot)$ , in terms of the existence of a common Lyapunov function. In particular, our results apply to switched dynamical systems such as (1), involving (possibly unbounded) operators on a Banach space  $X$ .

We have shown that the existence of a norm, decaying uniformly along trajectories and either bounded from above by a multiple of the Banach norm in presence of a uniform exponential growth bound for the switched system or comparable with the Banach norm is equivalent to the switched system being globally uniformly exponentially stable. The latter generalizes a well-known result for switched linear dynamical systems in  $\mathbb{R}^n$ ,  $n \in \mathbb{N}$  [17]. In the case in which  $X$  is a separable Hilbert space the common Lyapunov function is shown to be Fréchet directionally differentiable.

As an application, our results answer for example the question of existence of a common Lyapunov function for the switched linear hyperbolic system with reflecting boundaries considered in [1].

Concerning the differences between Theorems 3 and 6, we already noticed that a Lyapunov function  $V(\cdot)$  satisfying condition ii) of the statement of Theorem 3

does not necessarily satisfy the stronger conditions appearing in the statement of Theorem 6. Hence, Theorem 3 is better suited for proving the global uniform exponential stability of a switched system (although the uniform exponential growth boundedness should also be proved), while Theorem 6 provides more information on a switched system that is known to be globally uniformly exponentially stable, by tightening the properties satisfied by  $V(\cdot)$ .

In the same spirit one could characterize the global uniform exponential stability of a switched system by (a priori) loosening the hypotheses on  $V(\cdot)$ , replacing inequalities (28) and (43) by

$$L_j V(x) \leq -\kappa \|x\|_X^2, \quad \text{for all } j \in Q, \text{ for some } \kappa > 0.$$

One could also remove the hypothesis that  $V$  is the square of a norm. Conversely, one could (a priori) tighten the same hypotheses replacing  $L_j V(x)$  by

$$\bar{L}_j V(x) = \limsup_{t \downarrow 0} \frac{V(T_j(t)x) - V(x)}{t}.$$

As an open problem it remains to understand if smoothing can improve the regularity properties of the Lyapunov function  $V(\cdot)$ . For the finite-dimensional case, it is for instance known that  $V(\cdot)$  can be taken polyhedral or polynomial [2, 3, 8]. It would be interesting to recover results in the same direction for infinite dimensional switched systems.

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